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A SOLUTION OF THE FINITE-DIMENSIONAL HOMOGENEOUS BANACH SPACE PROBLEM

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ABSTRACT

It is proved that if a finite-dimensional Banach space X has the property that all its αn-dimensional subspaces are K-isomorphic, for some $0 < \alpha < 1$ and $K \geq 1$, then X is $f(\alpha, K)$-isomorphic to a Hilbert space, where $f(\alpha, K)$ is $cK^{3/2}$, if $0 < \alpha < 2/3$ and cK^2, if $2/3 < \alpha < 1$, and where $c = c(\alpha)$ depends on α only.

Introduction

The problem of infinite-dimensional homogeneous Banach spaces has been around since the very beginning of the theory. S. Banach in the original Polish edition of his fundamental book [Ba] attributed this problem to S. Mazur (p. 227). Despite the fact that more than half a century has elapsed, relatively little is known about the structure of such spaces. A local finite-dimensional version of the problem was raised by V. Milman. It can be stated as follows.

(Q) *Given $\alpha \in (0,1)$ and $K > 1$, does there exist a function $f(\alpha, K)$ with the property that whenever X is an n-dimensional Banach space such that all its $[\alpha n]$-dimensional subspaces are K-isomorphic then X is $f(\alpha, K)$-isomorphic to the n-dimensional Hilbert space l_n^2 ?*

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J. Bourgain proved in [B.2] that the answer to the question above is positive for α sufficiently small. His proof yields a rather unpleasant function $f(\alpha, K)$. The aim of this paper is to present a complete solution to (Q) for all $\alpha \in (0,1)$ and $K > 1$; furthermore, we give a simple, although probably not asymptotically best, dependence of f on α and K. The general scheme of the proof follows similar lines to Bourgain's. For an isometric variant of the homogeneous spaces problem consult Gromov's paper [Gr]; some partial results on the infinite-dimensional version can be found in Johnson's paper [J].

Let us describe the main steps of the proof and the content of the paper. For technical reasons we work in the dual setting, with quotients rather than subspaces. In the first step, Section 2, we study random quotients of a fixed n-dimensional Banach space $X = (\mathbb{R}^n, \|\cdot\|)$ and we generalize the fundamental idea of Gluskin [G.1] for arbitrary spaces. Let us mention that an investigation of random quotients of l_n^1 (and of l_n^p) proved fruitful in several long-standing problems on geometry of Banach spaces (cf. e.g. [B.1], [G.2], [Ma.2], [Ma.3], [R], [Sz.2]). Here we give lower estimates for the Banach–Mazur distance between random $[\alpha n]$-dimensional quotients of X in terms of some geometric and volumetric invariants of X. The crucial point of this approach is stated in Proposition 2.2 which contains rather delicate estimates of Haar measure of some subsets of Grassmann manifold $G_{n,\alpha n}$, related to a fixed operator acting in $\mathbb{R}^n$. The proof of this proposition is deferred to Section 5.

The next important step is to show that by first passing to a suitable quotient Y of X and only then considering random $[\alpha n]$-dimensional quotients of Y, one can essentially improve lower estimates for mutual distances of these random quotients. This is achieved by two results from Section 3. Proposition 3.2 establishes an inequality between an invariant introduced in Section 2 and the volume ratio of a space. (This already leads to a generalization of Gluskin's theorem, in Theorem 3.6.) Further, Proposition 3.5 shows that every n-dimensional Banach space X has, for every $\delta \in (0,1)$, a $[\delta n]$-dimensional quotient Y with a "nicely bounded" volume ratio and such that the formal identity operator from $l_{[\delta n]}^1$ into $l_{[\delta n]}^2$ admits a "nicely bounded" factorization through Y. The proof of the latter result is based on a deep result of Milman [Mi], and recent techniques on Dvoretzky–Rogers factorization by Bourgain, Szarek and Talagrand [B-Sz] and [Sz-Ta]. Some other technical estimates of this section also require a recent refinement of Milman's result due to Pisier [P.1] and [P.2].

Section 4 contains the solution of the finite-dimensional homogeneous Banach spaces problem. Also, we give some other related results. In particular, we establish a dichotomous behaviour of the Banach–Mazur distance for families of random quotients of a finite-dimensional Banach space. It is shown that every space X has a quotient X_1 of proportional dimension, such that either X_1 is Euclidean, or, for a random pair of quotients of X_1, the Banach–Mazur distance is large.

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1. Notation and Terminology

Our notation essentially follows [M-S], [P.2] and [T], and all notions unexplained here can be found in one of these books. Some of frequently used general conventions and notions are described below.

The natural Euclidean norm on $\mathbb{R}^n$ is denoted by $\|\cdot\|_2$, so that $(\mathbb{R}^n, \|\cdot\|_2)$ is just l_n^2. The Euclidean unit ball is denoted by B_n^2 and $\{e_1, \dots, e_n\}$ is the standard unit vector basis in $\mathbb{R}^n$. The Euclidean unit sphere in $\mathbb{R}^n$ (resp. in a subspace E of $\mathbb{R}^n$) is denoted by S^{n-1} (resp. S_E), the rotation invariant normalized measure on S^{n-1} (resp. S_E) is denoted by $\mu_{S^{n-1}}$ (resp. μ_{S_E}).

Let $1 \le l \le n$. Denote by $G_{n,l}$ the Grassmann manifold of all l-dimensional subspaces of $\mathbb{R}^n$ with the Haar measure $h_{n,l}$. Fix $0 < \gamma < 1$. For a family of properties $\mathcal{M} = \{\mathcal{M}^n\}$ of γn-dimensional spaces we say that $\mathcal{M}$ is satisfied for a random subspace, if there exists $0 < \delta < 1$ such that

$$h_{n,\gamma n}\{E \in G_{n,\gamma n} \mid E \text{ has } \mathcal{M}_n\} \ge 1 - \delta^n,$$

for every $n = 1, 2, \dots$. The notion of a random quotient is defined by an analogous condition via "the duality between subspaces and quotients". For detailed information about random phenomena and related measure concentration phenomena we refer the reader to [M-S].

For an integer m, by $\mathcal{O}(m)$ we denote the orthogonal group acting on $\mathbb{R}^m$ and, by h_m, the normalized Haar measure on $\mathcal{O}(m)$. In the integral notation we shall write dU instead of $dh_m(U)$. If $\mathcal{G}$ is a subgroup of $\mathcal{O}(m)$ then, unless stated otherwise, we denote by $h_{\mathcal{G}}$ the normalized Haar measure on $\mathcal{G}$, and we shall use the integral notation $d_{\mathcal{G}}W$ for $dh^{\mathcal{G}}(W)$.

Let $(X, \|\cdot\|_X)$ be a finite-dimensional Banach space. The unit ball of X is denoted by B^X. Let $|\cdot|_2$ be a Euclidean norm on X and let $(\cdot,\cdot)$ be the associated inner product. For an orthogonal projection P on $(X, |\cdot|_2)$, consider the norm $\|\cdot\|_{P(X)}$ on $P(X)$ defined by

$$(1.1) \qquad \|y\|_{P(X)} = \inf\{\|y+z\|_X \mid z \in \ker P\} \quad \text{for} \quad y \in P(X).$$

Clearly, $(P(X), \|\cdot\|_{P(X)})$ is a quotient space of X, with the quotient map P, and conversely, every quotient of X can be identified with a space of this form.

If X and Y are Banach spaces with $\dim X = \dim Y$ then the (Banach–Mazur) distance between X and Y is given by

$$d(X,Y) = \inf\{\|T\|\,\|T^{-1}\| \mid T : X \to Y \text{ is an isomorphism}\}.$$

For any linear operator $u : l_n^2 \to X$, the set $u(B_n^2)$ is called an ellipsoid on X. Any ellipsoid on X determines a Euclidean norm on X, for which it serves as a unit ball. Conversely, any Euclidean norm on X is obtained this way.

We will often need the concept of volume in finite-dimensional Banach spaces, with respect to a given inner product. The volume of a convex body K in such a space X is defined by the identification of X with $\mathbf{R}^n$ via the inner product and by the normalized Lebesgue measure of K under this identification. Sometimes we will write $\mathrm{vol}_n K$, to emphasize the dimension of the space. Clearly, for two convex bodies $K_1, K_2 \subset X$, the ratio of volumes, $\mathrm{vol}\, K_1 / \mathrm{vol}\, K_2$, does not depend on the initial choice of the inner product. For more detailed information on volumes, related inequalities and their use in the Banach space theory we refer the reader to [P.2].

It is well-known (cf. e.g. [T], Section 15) that for any finite-dimensional Banach space X there is the (unique) ellipsoid $\mathcal{E} \subset B_X$ which has maximal volume among all ellipsoids contained in B_X, and there is a (unique) ellipsoid $\mathcal{E}' \supset B_X$ which has minimal volume among all ellipsoids containing B_X (the so-called F. John ellipsoids).

Throughout the paper, by a, a_1, etc. we denote universal constants, and by $c = c(s_1, s_2, \dots)$, $c_1 = c_1(s_1, s_2, \dots)$, etc. constants which depend on parameters $s_1, s_2, \dots$. Constants of both types may vary from context to context. Since all the results of the paper are about isomorphic and asymptotic behaviour, we may (and shall) adopt the convention that numbers like αn, βn, etc., when representing dimensions of linear spaces, are positive integers.

All the results of this paper can be carried over to the complex case. It requires some standard modifications such as identifying complex n-dimensional bodies with $2n$-dimensional real bodies, and replacing the orthogonal group $\mathcal{O}(n)$ by the unitary group $\mathcal{U}(n)$.

2. Distances Between Random Quotients Finite Dimensional Banach Spaces

In this section we discuss the space $\mathbf{R}^n$ with an arbitrary norm $\|\cdot\|$ and we denote this space by X. To emphasize the dimension, the unit ball of X is denoted by

B_n. Recall that the norm from l_n^2 is denoted by $\|\cdot\|_2$, the Euclidean unit ball is denoted by B_n^2 and $\{e_1,\dots,e_n\}$ denotes the standard unit vector basis in $\mathbb{R}^n$. For any Banach space $Y=(\mathbb{R}^n,\|\cdot\|_Y)$, $i_{X,Y}: X\to Y$ denotes the formal identity operator. If $X=l_n^p$ (resp. $Y=l_n^p$), we denote $i_{X,Y}$ by $i_{p,Y}$ (resp. $i_{X,p}$).

Let $1\le k\le n$. By $\mathbb{R}^k\subset\mathbb{R}^n$ we denote $\operatorname{span}\{e_1,\dots,e_k\}$. By $L(\mathbb{R}^k)$ we denote the space of all linear operators on $\mathbb{R}^k$. For every $(n-k)$-dimensional subspace E of $\mathbb{R}^n$ by P_E we denote the orthogonal projection on $E^\perp$. For every $(n-k)$-dimensional subspace E of $\mathbb{R}^n$, which does not intersect $\mathbb{R}^k$, by $Q_E:\mathbb{R}^n\to\mathbb{R}^k$ we denote the projection onto $\mathbb{R}^k$ with the kernel E.

By $Q_E(X)$ we denote $\mathbb{R}^k$ with the norm for which the unit ball is $Q_E(B_n)$. That is, $\|y\|_{Q_E(X)}=\inf\{\|y+z\|\mid z\in E\}$, for $y\in\mathbb{R}^k$. In particular, in this section we will identify the quotient space X/E (where $E\cap\mathbb{R}^k=\{0\}$) with $Q_E(X)$.

For $1\le l\le k\le n$ set

$$V_{l,k}=V_{l,k}(X)=\sup_{E\in G_{n,n-k}}\ \inf_{\substack{F\in G_{n,n-l}\\ F\supset E}}\left(\operatorname{vol}_l P_F(B_n)/\operatorname{vol}_l P_F(B_n^2)\right)^{1/l}. \tag{2.1}$$

Also, for $E\in G_{n,n-k}$ set

$$f(E)=\left(\frac{\operatorname{vol}_{k^2}\{T\mid\|T:l_k^1\to Q_E(X)\|\le1\}}{\operatorname{vol}_{k^2}\{T\mid\|T:l_k^2\to Q_E(X)\|\le1\}}\right)^{1/k^2}.$$

Define

$$W_k=W_k(X)=\sup_{E\in G_{n,n-k}}f(E). \tag{2.2}$$

The quantity $V_{l,k}(X)$ is closely related to the notion of the volume ratio numbers of operators, which will be of importance later on. Recall ([Mas]) that if $Y=(\mathbb{R}^n,\|\cdot\|_Y)$ then for $u:X\to Y$ and $1\le j\le n$, we set

$$\operatorname{vr}_j(u)=\sup_{E\in G_{n,n-j}}\left(\operatorname{vol}_j(P_E uB_X)/\operatorname{vol}_j P_E B_Y\right)^{1/j}. \tag{2.3}$$

For $Y=l_n^2$ the above definition coincides with the one introduced in [M-P.2] and discussed also in [P-T]. It is easy to see that

$$V_{l,k}(X)^{-1}=\inf_{E\in G_{n,n-k}}\operatorname{vr}_l(P_E i_{2,X}\mid E^\perp), \tag{2.4}$$

where $P_E i_{2,X}\mid E^\perp$ is an operator acting from $(E^\perp,\|\cdot\|_2)$ to the quotient $P_E(X)=(E^\perp,\|\cdot\|_{P_E(X)})$. Moreover, $V_{k,k}(X)=\operatorname{vr}_k(i_{X,2})$, so in particular, $V_{m,m}(X)\le V_{k,k}(X)$, for $1\le k\le m\le n$ (cf. [P-T]).

Finally, we require an invariant related to so-called Dvoretzky–Rogers factorization:

(2.5) $$\kappa = \kappa(X) = \|i_{1,X} : l_n^1 \to X\| \, \|i_{X,2} : X \to l_n^2\|.$$

The following fact is a reformulation of one of the main technical tools in [Sz.1].

FACT I: *Let $0 < \gamma < \alpha < 1$. There is $c_1 = c_1(\alpha, \gamma)$ such that a random subspace $E \in G_{n,(1-\alpha)n}$ has the property:*

(A) *The operator $Q_E \,|\, E^\perp : E^\perp \to \mathbb{R}^{\alpha n}$ has at most γn s-numbers larger than c_1.*

As an immediate consequence of property (A) one has

LEMMA 2.1: *Let $E \in G_{n,(1-\alpha)n}$ satisfy (A). There exists an orthogonal projection $P' \in L(\mathbb{R}^{\alpha n})$ with rank $P' \geq (\alpha - \gamma)n$ such that*

$$\|P' Q_E : l_n^2 \to l_{\alpha n}^2\| \leq c_1,$$

where $c_1 = c_1(\alpha, \gamma)$ is a constant from (A).

FACT II: *Let $0 < \gamma < \alpha < 1$. There is $c_2 = c_2(\alpha)$ and $\varepsilon = \varepsilon(\gamma) > 0$ such that a random subspace $E \in G_{n,(1-\alpha)n}$ has the property:*

(B) $\|Q_E e_i\|_2 \leq c_2$, for $i = \alpha n + 1, \alpha n + 2, \ldots, n$.

(C) dist $(Q_E e_i,$ span $\{Q_E e_k \mid (1-\gamma)n \leq k < i\}) > \varepsilon$, for $(1-\gamma)n < i \leq n$.

The standard proof based on the measure concentration phenomenon (cf. [M-S]) is left to the reader.

The above facts imply that the set

(2.6) $$\mathcal{F} = \{E \in G_{n,(1-\alpha)n} \mid E \text{ satisfies (A), (B), (C)}\}$$

has the measure

(2.7) $$h_{n,(1-\alpha)n}(\mathcal{F}) \geq 1 - \delta^n,$$

for some $0 < \delta(\alpha) < 1$.

Let $E \in G_{n,(1-\alpha)n}$ and $E \cap \mathbb{R}^k = \{0\}$. Consider the projection Q_E. Since Q_E is the identity on $\mathbb{R}^{\alpha n}$ then it is fully determined by vectors $Q_E e_i$ for $i > \alpha n$. Let us state the following obvious fact.

FACT III: *The distribution of sets* $\{Q_E e_i \mid \alpha n < i \leq n\} \in \mathbb{R}^{\alpha n} \times \cdots \times \mathbb{R}^{\alpha n}$ *is rotation invariant. That is,*
(D) *Let* $D_i \subset \mathbb{R}^{\alpha n}$, *for* $\alpha n < i \leq n$, *and let* $V \in \mathcal{O}(n)$. *Then*

$$\begin{aligned} &h_{n,(1-\alpha)n}\left(\{E \in G_{n,(1-\alpha)n} \mid Q_{V(E)}e_i \in D_i \text{ for } \alpha n < i \leq n\}\right) \\ =&h_{n,(1-\alpha)n}\left(\{E \in G_{n,(1-\alpha)n} \mid Q_E e_i \in D_i \text{ for } \alpha n < i \leq n\}\right). \end{aligned}$$

Random constructions presented in this section depend on measure estimates done for a fixed single operator. They are stated in the technical proposition which follows. These estimates depend on parameters α, β and β' satisfying $0 < \beta < \beta' < \alpha < 1$ and we present below two most important versions. The first admits β and β' arbitrarily close to α; in this case however we require that the operator has only relatively few s-numbers smaller than or equal to 1. The second version works for an operator which has many (up to a half) s-numbers smaller than or equal to 1, but then β and β' are required to be smaller than $\alpha/2$.

PROPOSITION 2.2: *Let X be an n-dimensional Banach space. Let $0 < \alpha < 1$ and let $A > 0$. Let $E_0 \in G_{n,(1-\alpha)n}$, with $E_0 \cap \mathbb{R}^k = \{0\}$, satisfy (A) and let $Y_{\alpha n} = Q_{E_0}(X)$.*
(i)
Let $0 < \beta < \beta' < \alpha$ and let $\gamma = \min\left((\alpha - \beta')/4, 1-\alpha\right)$. There exist $c = c(\alpha, \beta, \beta')$ such that for every operator $T \in L(\mathbb{R}^{\alpha n})$ which has at least $(1-2\gamma)n$ s-numbers larger than or equal to 1 we have

$$h_{n,(1-\alpha)n}\left(\{E \in \mathcal{F} \mid \|TQ_E : l_n^1 \to Y_{\alpha n}\| \leq A\}\right) \leq \left(cAV_{\beta n,\beta' n}\right)^{\gamma\beta n^2}. \tag{2.8}$$

(ii)
Let $0 < \beta < \beta' < \alpha/2$ and let $\gamma = \min\left((\alpha/2 - \beta')/2, 1-\alpha\right)$. There exist $c = c(\alpha, \beta, \beta')$ such that (2.8) holds for every operator $T \in L(\mathbb{R}^{\alpha n})$ which has at least $\alpha n/2$ s-numbers larger than or equal to 1.

We postpone the proof of the proposition to the last section.

The following lemma is a standard ingredient.

LEMMA 2.3: *Let X be an n-dimensional Banach space. Let $0 < \gamma < \alpha < 1$. Let $E_0 \in G_{n,(1-\alpha)n}$ satisfy (A) and let $Y_{\alpha n} = Q_{E_0}(X)$. For $A > 0$ set*

$$\begin{aligned} \mathcal{T}_A = \{T \in L(\mathbb{R}^{\alpha n}) \mid \|T : l_{\alpha n}^1 \to Y_{\alpha n}\| \leq A \text{ and} \\ T \text{ has at most } 2\gamma n \text{ } s\text{-numbers} < 1\}. \end{aligned}$$

Then for any $b > 0$, $\mathcal{T}_A$ admits a bA-net $\mathcal{N}$ in the operator norm from $l^2_{\alpha n}$ into $Y_{\alpha n}$, with the cardinality

$$|\mathcal{N}| \le (W_{\alpha n}/b)^{(\alpha n)^2}.$$

The standard proof of the lemma is based on the comparison of volumes argument (cf. e.g. [G.1], also [T] Section 38) and follows directly from the definition of $W_{\alpha n}$.

THEOREM 2.4: *Let X be an n-dimensional Banach space. Let $0 < \alpha < 1$. Let $E_0 \in G_{n,(1-\alpha)n}$ satisfy (A) and let $Y_{\alpha n} = Q_{E_0}(X)$.*
(i) Let $0 < \beta < \beta' < \alpha$ and let $\gamma = \min((\alpha - \beta')/4, 1-\alpha)$. A random subspace $E \in G_{n,(1-\alpha)n}$ has the property: for every operator $T \in L(\mathbb{R}^{\alpha n})$ which has at most $2\gamma n$ s-numbers smaller than 1 we have

$$\|TQ_E : l^1_n \to Y_{\alpha n}\| \ge c W_{\alpha n}^{-\alpha^2/\gamma\beta} V^{-1}_{\beta n,\beta' n}, \tag{2.9}$$

where $c = c(\alpha, \beta, \beta') > 0$.
(ii) Let $0 < \beta < \beta' < \alpha/2$ and let $\gamma = \min((\alpha/2 - \beta')/2, 1-\alpha)$. Then a random subspace $E \in G_{n,(1-\alpha)n}$ satisfies (2.9) for every operator $T \in L(\mathbb{R}^{\alpha n})$ which has at most $\alpha n/2$ s-numbers smaller than 1.

Proof: Set

$$A = \frac{1}{2}(2c_2 W_{\alpha n})^{-\alpha^2/\gamma\beta}(c_3 V_{\beta n,\beta' n})^{-1},$$

where c_2 and c_3 are constants from (B) and Proposition 2.2 respectively. Let $\mathcal{T}_A$ be the set of operators defined in Lemma 2.3. For $T \in \mathcal{T}_A$ let $\mathcal{K}_T$ be the set from the conclusion of Proposition 2.2. Let $\mathcal{N}$ be an $(A/2c_2)$-net in $\mathcal{T}_A$ of minimal cardinality.

For arbitrary $E \in \mathcal{F}$ we have, by (B), $Q_E(B^1_n) \subset c_2 B^2_{\alpha n}$. Therefore, if $T \in L(\mathbb{R}^{\alpha n})$ then

$$\|TQ_E : l^1_n \to Y_{\alpha n}\| \le c_2 \|T : l^2_{\alpha n} \to Y_{\alpha n}\|.$$

Thus for every $T \in \mathcal{T}_A$ there is $T' \in \mathcal{N}$ such that

$$\begin{aligned}\|TQ_E : l^1_n \to Y_{\alpha n}\| &\ge \|T'Q_E : l^1_n \to Y_{\alpha n}\| - \|(T-T')Q_E : l^1_n \to Y_{\alpha n}\| \\ &\ge \|T'Q_E : l^1_n \to Y_{\alpha n}\| - c_2\|(T-T') : l^2_{\alpha n} \to Y_{\alpha n}\| \\ &\ge \|T'Q_E : l^1_n \to Y_{\alpha n}\| - A/2.\end{aligned}$$

Fix $E \in \mathcal{F} \setminus \bigcup_{T \in \mathcal{N}} \mathcal{K}_T$. The previous calculation shows that

$$\|TQ_E : l_n^1 \to Y_{\alpha n}\| \geq A/2,$$

for every $T \in \mathcal{T}_A$. Also, if $T \notin \mathcal{T}_A$ then $\|T : l_{\alpha n}^1 \to Y_{\alpha n}\| > A$ and hence

$$\|TQ_E : l_n^1 \to Y_{\alpha n}\| > A.$$

Finally observe that (2.7), (2.8) and Lemma 2.3 easily imply

$$h_{n,(1-\alpha)n}\left(\mathcal{F} \setminus \bigcup_{T \in \mathcal{N}} \mathcal{K}_T\right) \geq 1 - \delta^n - (1/2)^{\gamma\beta n^2},$$

for some $0 < \delta < 1$, which completes the proof. □

Our next step will be to consider operators acting between (different) quotients of X. The norms of such operators do not depend on a specific (isometric) representation of X in $\mathbb{R}^n$, while the parameter $V_{\beta n, \beta' n}(X)$ does. This yields a necessity of a normalization condition. The most convenient one is $\|i_{X,2}\| = \|i_{X,2} : X \to l_n^2\| = 1$. Note that with this normalization we have $\kappa(X) = \|i_{1,X} : l_n^1 \to X\|$. The following theorem is an immediate consequence of

THEOREM 2.5: *Let X be an n-dimensional Banach space with $\|i_{X,2}\| = 1$ and let $0 < \alpha < 1$. Let $E_0 \in G_{n,(1-\alpha)n}$ satisfy (A) and let $Y_{\alpha n} = Q_{E_0}(X)$.*
(i) Let $0 < \beta < \beta' < \alpha$ and let $\gamma = \min((\alpha - \beta')/4, 1 - \alpha)$. A random subspace $E \in G_{n,(1-\alpha)n}$ has the property: for every operator $T \in L(\mathbb{R}^{\alpha n})$ which has at most $2\gamma n$ s-numbers smaller than 1 we have

$$\|TQ_E : X \to Y_{\alpha n}\| \geq c\kappa(X)^{-1} W_{\alpha n}^{-\alpha^2/\gamma\beta} V_{\beta n, \beta' n}^{-1}, \tag{2.10}$$

where $c = c(\alpha, \beta, \beta') > 0$.
(ii) Let $0 < \beta < \beta' < \alpha/2$ and let $\gamma = \min((\alpha/2 - \beta')/2, 1 - \alpha)$. Then a random subspace $E \in G_{n,(1-\alpha)n}$ satisfies (2.10) for every operator $T \in L(\mathbb{R}^{\alpha n})$ which has at most $\alpha n/2$ s-numbers smaller than 1.

The results we proved so far can be applied for estimating the distance between random αn-dimensional quotients of an n-dimensional space X. We prove two different versions of such estimates. We start with a theorem of general interest which in many particular cases gives the asymptotically best lower estimate.

THEOREM 2.6: *Let X be an n-dimensional Banach space with $\|i_{X,2}\| = 1$. Let $0 < \beta < \beta' < \alpha/2 < 1/2$ and let $\gamma = \min((\alpha/2 - \beta')/2, 1-\alpha)$. There exist $E_1,\ E_2 \in G_{n,(1-\alpha)n}$ such that*

$$d(Q_{E_1}(X), Q_{E_2}(X)) \geq c\kappa(X)^{-2} W_{\alpha n}^{-2\alpha^2/\gamma\beta} V_{\beta n,\beta' n}^{-2}, \tag{2.11}$$

where $c = c(\alpha, \beta, \beta') > 0$. In fact, (2.11) is satisfied for random subspaces $E_1,\ E_2 \in G_{n,(1-\alpha)n}$.

Proof: Theorem 2.5 (ii) implies that for any $i,j = 1,2,\ i \neq j$, the subset $\mathcal{B}_{i,j}$ of $G_{n,(1-\alpha)n} \times G_{n,(1-\alpha)n}$ defined by

$$\begin{aligned}\mathcal{B}_{i,j} = \{(E_i, E_j) |\ \ & \|T : Q_{E_i}(X) \to Q_{E_j}(X)\| \geq cW_{\alpha n}^{-\alpha^2/\gamma\beta} V_{\beta n,\beta' n}^{-1} \\ & \text{for every } T \in L(\mathbb{R}^{\alpha n}) \text{ which has} \\ & \text{at most } \alpha n/2\ s\text{-numbers } < 1\}\end{aligned}$$

has measure larger than $1 - \delta^n$, for some $0 < \delta < 1$. Clearly any isomorphism $T \in L(\mathbb{R}^{\alpha n})$ can be normalized so that both T and T^{-1} have at most $\alpha n/2$ s-numbers < 1. Therefore, by Theorem 2.5, for any $(E_1, E_2) \in \mathcal{B}_{1,2} \cap \mathcal{B}_{2,1}$ and any T, both norms $\|T : Q_{E_1}(X) \to Q_{E_2}(X)\|$ and $\|T^{-1} : Q_{E_2}(X) \to Q_{E_1}(X)\|$ admit suitable lower estimates. This obviously concludes the proof. □

We pass now to the distance estimate in terms of arbitrary $\beta < \beta' < \alpha$, which is important for further applications. In this case we cannot apply Theorem 2.5 simultaneously for both T and T^{-1}. The control of the norm of one of these operators is achieved by the following simple lemma.

LEMMA 2.7: *Let X be an n-dimensional Banach space and let $0 < \gamma < \alpha < 1$. Let $E_1,\ E_2 \in \mathcal{F}$. For every operator $T \in L(\mathbb{R}^{\alpha n})$ which has at least $2\gamma n$ s-numbers larger than or equal to 1 we have*

$$\|T : Q_{E_1}(X) \to Q_{E_2}(X)\| \geq c\kappa(X)^{-1},$$

where $c = c(\alpha, \gamma) > 0$.

Proof: Since E_2 satisfies (A), Lemma 2.1 yields that for some orthogonal projection $P \in L(\mathbb{R}^{\alpha n})$ with rank $P \geq (\alpha - \gamma)n$ we have

$$\|PQ_{E_2} : l_n^2 \to l_{\alpha n}^2\| \leq c_1,$$

where c_1 is as in (A).

Fix $T \in L(\mathbb{R}^{\alpha n})$ with at least $2\gamma n$ s-numbers larger than or equal to 1. Then we have

$$\|T : Q_{E_1}(X) \to Q_{E_2}(X)\| \geq c\kappa(X)^{-1}\|PQ_{E_2}T : l^1_{\alpha n} \to l^2_{\alpha n}\|, \tag{2.12}$$

where $c = c(\alpha, \gamma) = (c_1)^{-1}$.

Denote the operator $PQ_{E_2}T$ by R. Observe that R has at least γn s-numbers larger than or equal to 1, and so its Hilbert–Schmidt norm satisfies $HS(R) \geq (\gamma n)^{1/2}$. Thus

$$\begin{aligned} \|R : l^1_{\alpha n} \to l^2_{\alpha n}\| &= \max\{\|Re_i\|_2 \mid i = 1, \cdots, \alpha n\} \\ &\geq (\alpha n)^{-1/2}\left(\sum_{i=1}^{\alpha n}\|Re_i\|_2^2\right)^{1/2} \\ &= (\alpha n)^{-1/2}HS(R) \geq (\gamma/\alpha)^{1/2}. \end{aligned} \tag{2.13}$$

We complete the proof combining the last estimate with (2.12). □

THEOREM 2.8: *Let X be an n-dimensional Banach space with $\|i_{X,2}\| = 1$. Let $0 < \beta < \beta' < \alpha < 1$ and let $\gamma = \min((\alpha - \beta')/4, 1 - \alpha)$. There exist $E_1, E_2 \in G_{n,(1-\alpha)n}$ such that*

$$d(Q_{E_1}(X), Q_{E_2}(X)) \geq c\kappa(X)^{-2}W_{\alpha n}^{-\alpha^2/\gamma\beta}V^{-1}_{\beta n, \beta' n}, \tag{2.14}$$

where $c = c(\alpha, \beta, \beta') > 0$. In fact, (2.14) is satisfied for random subspaces $E_1, E_2 \in G_{n,(1-\alpha)n}$.

An easy proof based on the previous lemma and Theorem 2.5 is left to the reader.

3. Controlling $\kappa(X)$ and $W_{\alpha n}(X)$. The Original Gluskin Theorem

In Section 2 we considered a space $X = (\mathbb{R}^n, \|\cdot\|)$, together with the fixed (natural) Euclidean norm. This norm was also used to define all parameters of X involved in the estimates. To consider corresponding parameters for an arbitrary n-dimensional Banach space X, we need to identify the space with $\mathbb{R}^n$, or, what amounts to the same, to define on X a Euclidean norm. The aim of this section is to show that given a finite-dimensional Banach space, a suitable proportional dimensional quotient admits a Euclidean norm for which the parameters $\kappa(\cdot)$ and $W_k(\cdot)$ admit universal upper bounds. This will allow us to eliminate $\kappa(\cdot)$ and $W_k(\cdot)$ from the lower estimates of Section 2, and get meaningful inequalities for a large class of finite-dimensional Banach spaces.

Results of this section are based on a combination of recent deep techniques in the local theory of Banach spaces: the approach of Milman for finding large quotients with bounded volume ratio, and a random selection method for finding large quotients with bounded κ, developed by Bourgain, Szarek and Talagrand.

We recall the notion of the volume ratio, introduced in [Sz-T]. Let Y be a k-dimensional Banach space with the unit ball B_Y. Let $\mathcal{E} \subset B_Y$ be the ellipsoid of maximal volume contained in B_Y. The volume ratio of Y, $\mathrm{vr}(Y)$, is defined by

$$\mathrm{vr}\,(Y) = (\text{ vol } B_Y / \text{ vol } \mathcal{E})^{1/k}. \tag{3.1}$$

The following lemma is well known to specialists.

Lemma 3.1: *Let Y be a k-dimensional Banach space, let $1 \le l \le k$ and let Z be an l-dimensional quotient of Y. Then*

$$\mathrm{vr}\,(Z) \le \binom{k}{l}^{1/l} \mathrm{vr}\,(Y)^{k/l}.$$

Proof: Let $\mathcal{E} \subset B_Y$ be the ellipsoid of maximal volume contained in B_Y. Let $Q : Y \to Z$ denote the quotient map. Without loss of generality we may identify Q with an orthogonal projection in Y (in the Euclidean structure given by $\mathcal{E}$) and Z with the range of Q. Denote $\ker Q = E$. Then we have (cf. e.g. [P.2], Chapter 8)

$$\left(\frac{\text{vol } Q(B_Y)}{\text{vol } Q(\mathcal{E})}\right) \times \left(\frac{\text{vol } (E \cap B_Y)}{\text{vol } (E \cap \mathcal{E})}\right) \le \binom{k}{l} \left(\frac{\text{vol } B_Y}{\text{vol } \mathcal{E}}\right). \tag{3.2}$$

Since obviously $E \cap \mathcal{E} \subset E \cap B_Y$, by (3.2) and (3.1) we get

$$\mathrm{vr}\ (Z) \le \left(\frac{\text{vol } Q(B_Y)}{\text{vol } Q(\mathcal{E})}\right)^{1/l} \le \binom{k}{l}^{1/l} \left(\frac{\text{vol } B_Y}{\text{vol } \mathcal{E}}\right)^{1/l}, \tag{3.3}$$

which completes the proof. □

The relevance of the bounded volume ratio property in the present context lies in the fact that this property allows a good control of the parameter W_k.

PROPOSITION 3.2: *Let Y be a k-dimensional Banach space and let $\|\cdot\|_2$ be an arbitrary Euclidean norm on Y. Let $1 \le l \le k$. Then*

$$W_l(Y) \le a\binom{k}{l}^{1/l} \mathrm{vr}\,(Y)^{k/l},$$

where $a \ge 1$ is a universal constant.

Proof: We shall show that any l-dimensional Banach space $F = (\mathbf{R}^l, \|\cdot\|)$ satisfies

$$\left(\frac{\mathrm{vol}\,\{T \mid \|T : l_l^1 \to F\| \le 1\}}{\mathrm{vol}\,\{T \mid \|T : l_l^2 \to F\| \le 1\}}\right)^{1/l^2} \le \mathrm{vr}\,(F). \tag{3.4}$$

This will imply the conclusion by the definition of $W_l(Y)$ and Lemma 3.1. To prove (3.4) observe first that

$$\mathrm{vol}\,\{T \mid \|T : l_k^1 \to F\| \le 1\} = (\,\mathrm{vol}\,B_F)^l.$$

On the other hand, let $\mathcal{E} \subset B_F$ be the ellipsoid of maximal volume contained in B_F, and let H denote F with the Euclidean structure determined by $\mathcal{E}$. Then

$$\{T \mid \|T : l_l^2 \to F\| \le 1\} \supset \{T \mid \|T : l_l^2 \to H\| \le 1\}.$$

Thus

$$\begin{aligned}\mathrm{vol}\,\{T \mid \|T : l_l^2 \to F\| \le 1\} &\ge \mathrm{vol}\,\{T \mid \|T : l_l^2 \to H\| \le 1\}\\ &= \left(\frac{\mathrm{vol}\,\mathcal{E}}{\mathrm{vol}\,B_l^2}\right)^l \mathrm{vol}\,\{T \mid \|T : l_l^2 \to l_l^2\| \le 1\}\\ &\ge (a'\,\mathrm{vol}\,\mathcal{E})^l,\end{aligned}$$

where $a' > 0$ is a universal constant. The last inequality follows from the well known estimate

$$\mathrm{vol}\,\{T \mid \|T : l_l^2 \to l_l^2\| \le 1\} \ge \left(a'\ \mathrm{vol}\,B_l^2\right)^l,$$

where $a' > 0$ is a universal constant (cf. e.g. [G.1], also [T] Section 38). Combining these estimates we get (3.4). □

In the sequel we shall use a fundamental result of Milman [Mi] in the important refined version due to Pisier [P.1]. It requires additional notions of Gelfand, Kolmogorov and entropy numbers.

Let X and Y be Banach spaces and let $u : X \to Y$ be an operator. For $m = 1, 2, \dots$ define the m-th Gelfand number by

$$c_m(u) = \inf\{\|u \mid E\| \mid E \subset X, \text{ codim } E < m\},$$

and the m-th Kolmogorov number by

$$d_m(u) = \inf\{\|Q_E u\| \mid Q_E : Y \to Y/E \text{ is the quotient map, } \dim E < m\}.$$

Define the m-th entropy number by

$$e_m(u) = \inf\left\{\varepsilon \mid \exists\{y_i\}_1^{2^{m-1}} \subset Y, \quad u(B_X) \subset \bigcup_{i=1}^{2^{m-1}} (y_i + \varepsilon B_Y)\right\}.$$

Moreover if $Y = l_n^2$ and $H \subset l_n^2$, by $v_H : H \to u^*(H)$ and $(u^{-1})_H : H \to u^{-1}(H)$ denote the restriction of u^* and u^{-1} respectively. Set

$$\tilde{c}_m(u) = \sup\{c_m((v_H)^*) \mid H \subset l_n^2\},$$
$$\tilde{d}_m(u^{-1}) = \sup\{d_m((u^{-1})_H) \mid H \subset l_n^2\},$$
$$\tilde{e}_m(u) = \sup\{e_m((v_H)^*) \mid H \subset l_n^2\}.$$

Now the result states.

THEOREM 3.3: *Let X be an n-dimensional Banach space. There exists an isomorphism $u : X \to l_n^2$ such that for all $1 \le m \le n$ one has*

$$d_m(u^{-1}) \le an/m \quad \text{and} \quad c_m(u) \le an/m. \tag{3.5}$$

In particular, $\max\left(e_m(u), e_m(u^{-1})\right) \le a'n/m$.

Moreover u can be taken to additionally satisfy

$$\max\left(\tilde{c}_m(u), \tilde{d}_m(u^{-1}), \tilde{e}_m(u)\right) \le a(n/m)^2 \quad \text{for } 1 \le m \le n. \tag{3.6}$$

Here $a > 0$, $a' > 0$ are universal constants.

The proof of Theorem 3.3 can be found in [P.1] and [P.2] Theorem 7.13. The moreover part is discussed (in the dual form) in [P.1] Corollary 2.8. Estimates for the entropy numbers follow from Carl's inequality (cf. [P.2]).

Further on we will require an inequality relating the existence of Euclidean sections of a given finite-dimensional Banach space to some volume estimates. In the preliminary version of this paper [M-T.1] we used Szarek's volume ratio result; and in [M-T.2] we used the inequality from [P-T]. M. Junge has recently informed us about the following strengthening of the latter inequality. Recall that the volume ratio numbers were defined in (2.3).

PROPOSITION 3.4: *Let $\zeta > 1$. Let X be an n-dimensional Banach space and let $w : l_2^n \to X$ be an operator. For every $1 \le m \le n/\zeta$ one has*

$$d_{\zeta m}(w) \le c(n/m)^2 \ \mathrm{vr}_m(w),$$

where $c = c(\zeta)$.

Proof: Let $u : X \to l_2^n$ be an isomorphism as in Theorem 3.3. Fix m and let $m' = (\zeta - 1)m$. Using well-known additivity property of Kolmogorov numbers and some properties of volume ratio numbers ([Mas], [P-T]) we get

$$\begin{aligned} d_{\zeta m}(w) &\le d_{m'}(u^{-1}) d_m(uw) \le d_{m'}(u^{-1}) \ \mathrm{vr}_m(uw) \\ &\le d_{m'}(u^{-1}) \ \mathrm{vr}_m(u) \ \mathrm{vr}_m(w) \le d_{m'}(u^{-1}) 2e_m(u) \ \mathrm{vr}_m(w) \\ &\le c(n/m)^2 \ \mathrm{vr}_m(w), \end{aligned}$$

completing the proof. □

Now we come to the main subject of this section which is concerned with quotients having simultaneously bounded volume ratio and uniformly bounded κ. This is described in the following proposition.

PROPOSITION 3.5: *Let X be an n-dimensional Banach space. Let $0 < \lambda < 1$. There exists a quotient of X, say Y, with $\dim Y = k \ge \lambda n$, and a Euclidean norm $|\cdot|_2$ on Y such that*

$$\|i_{X,2} : X \to (X, |\cdot|_2)\| = 1 \tag{3.7}$$

and

$$\mathrm{vr}(Y) \le C \quad \text{and} \quad \kappa(Y) \le C, \tag{3.8}$$

where $C = C(\lambda)$.

Moreover, if $\|\cdot\|_2$ denotes the Euclidean norm on X determined by the ellipsoid of minimal volume containing the unit ball of X, and if $Y = P(X)$, for some orthogonal projection P, then the norm $|\cdot|_2$ can be chosen to satisfy

$$(3.9) \qquad (1/c)\|y\|_2 \leq |y|_2 \leq c\|y\|_2 \quad \textit{for} \quad y \in Y,$$

where $c = c(\lambda)$.

Remark: The functions $C(\lambda)$ and $c(\lambda)$ tend to infinity, as $\lambda \to 1$. The proof below shows that they can be taken as polynomials in $(1-\lambda)^{-1}$.

The upper estimate for $\kappa(Y)$ stated in the proposition is clearly equivalent to the existence of an orthonormal basis $\{e_1, \dots, e_k\}$ in $(Y, |\cdot|_2)$ such that

$$(3.10) \qquad |y|_2 \leq \|y\| \leq C\sum_i |t_i| \quad \text{for} \quad y = \sum_i t_i e_i \in Y.$$

Proof: Set $\delta = \lambda^{1/3}$. We start by finding a quotient of X with a bounded volume ratio. Let $u : X \to l_n^2$ be an isomorphism constructed in Theorem 3.3 satisfying (3.6). Let $m = (1-\delta)n$. Then $d_m(u^{-1}) \leq c'$, where $c' = c'(\lambda)$. From the definition of d_m this means that there exists a quotient space of X, say F, with $\dim F = n - m$, and the quotient map $Q : X \to F$ such that $\|Qu^{-1}\| \leq d_m(u^{-1}) \leq c'$. Consider an ellipsoid on F defined by $\mathcal{E} = Qu^{-1}(B_n^2)$. Then

$$(3.11) \qquad (1/c')\mathcal{E} \subset Q(B_X) = B_F.$$

Moreover, consider the operator $w_F : F \to l_{n-m}^2$ such that $w_F Q = P_E u$, where $E = u(\ker Q)$ and P_E is the orthogonal projection with the kernel E. Then (3.6) implies that w_F satisfies

$$e_m(w_F) \leq a(n/m)^2 = c'',$$

where $c'' = c''(\lambda)$. A standard covering argument (cf. [P.2] (7.40) and (7.41)) shows that

$$(\text{vol } B_F/\text{vol } \mathcal{E})^{1/(n-m)} \leq 2e_m(w_F) \leq 2c''.$$

Combining with (3.11) we get

$$(3.12) \qquad \text{vr}\,(F) \leq (\text{ vol } B_F/\text{ vol } (1/c')\mathcal{E})^{1/(n-m)} \leq 2c'c'' = C.$$

Now, let $\|\cdot\|_2$ be the Euclidean norm on X determined by the ellipsoid of minimal volume containing the unit ball of X. As usual, we may assume that $F = Q(X)$, for some orthogonal projection Q. To prove the estimate for κ we need to construct further quotients.

Let us recall a well-known and important property of the minimal volume ellipsoid (cf. e.g. [T] Proposition 15.11). It says that for any orthogonal projection R of rank k we have

$$\|R : X \to (X, \|\cdot\|_2)\| \geq (k/n)^{1/2}. \tag{3.13}$$

In particular, if S is an orthogonal projection in $F = Q(X)$ of rank k, then

$$\|S : F \to (F, \|\cdot\|_2)\| \geq \|SQ : X \to (F, \|\cdot\|_2)\| \geq (k/n)^{1/2}.$$

This allows us to construct, by [Sz-Ta] Corollary 5, a sequence of s vectors $e_1, \dots, e_s$ in F, with $s = \delta \dim F = \delta^2 n$ such that $\|e_i\|_F = 1$, for $i = 1, \dots, s$ and

$$\sum_i |t_i| \geq \|\sum_i t_i e_i\|_2 \geq c''' \left(\sum_i |t_i|^2\right)^{1/2} \quad \text{for} \quad (t_i) \in \mathbf{R}^s, \tag{3.14}$$

where $c''' = c'''(\lambda) > 0$.

Let $Y =$ span $\{e_i\}_{i \leq s}$, with the quotient norm from F, say $\|\cdot\|_Y$, given by the orthogonal projection R from F onto Y. Clearly, $Y = RQ(X)$ is also a quotient of X. We have

$$\|y\|_2 = \|Ry\|_2 \leq \inf\{\|y + f\|_F \mid f \in \ker R\} = \|y\|_Y \quad \text{for} \quad y \in Y. \tag{3.15}$$

Moreover, $\|e_i\|_Y \leq 1$ for $i = 1, \dots, s$. Finally, define the norm $|\cdot|_2$ on Y by

$$\left|\sum_i t_i e_i\right|_2 = D \left(\sum_i |t_i|^2\right)^{1/2} \quad \text{for} \quad (t_i) \in \mathbf{R}^s, \tag{3.16}$$

where the normalization factor D is chosen so that (3.7) is satisfied. Combining with (3.14) and (3.15) we get (3.10). Finally, using Lemma 3.1 we complete (3.8).

Notice that (3.14) immediately yields that the norm $|\cdot|_2$ defined in (3.16) satisfies the upper estimate in (3.9). To get the norm satisfying also the lower estimate we need to replace, in the above construction, vectors (e_i) with vectors $e'_1, \ldots, e'_k$ such that

$$(1/c')\left(\sum_i |t_i|^2\right)^{1/2} \leq \|\sum_i t_i e'_i\|_2 \leq c'\left(\sum_i |t_i|^2\right)^{1/2}, \tag{3.17}$$

for all $(t_i) \in \mathbb{R}^k$, where $c' = c'(\lambda)$. The approach from [B-Sz], Lemmas C and D, shows that there exist an orthogonal projection $S : F \to F$ (with the norm $\|\cdot\|_2$) and a subset $\sigma \subset \{1, \ldots, s\}$ with $|\sigma| \geq \delta s = \lambda n$ such that the vectors $e'_i = Se_i$ for $i \in \sigma$ satisfy (3.17). The proof of the moreover part of the proposition is then concluded by setting $Y =$ span $\{e'_i\}_{i\in\sigma}$, with the quotient norm defined by the orthogonal projection and repeating the argument above. □

The following theorem is a generalization and an "abstract form" of the result of Gluskin [G.1].

THEOREM 3.6: *Let $0 < \alpha < 1$. Let X be an n-dimensional Banach space, and let $\|\cdot\|_2$ be the Euclidean norm on X determined by the ellipsoid of minimal volume containing the unit ball of X. For any $0 < \beta < \beta' < \alpha/2$, X has two αn-dimensional quotients, say F_1 and F_2, such that*

$$d(F_1, F_2) \geq c V^{-2}_{\beta n, \beta' n}(X), \tag{3.18}$$

where $V_{\beta n, \beta' n}$ is defined by (2.1) with respect to $\|\cdot\|_2$ and $c = c(\alpha, \beta, \beta') > 0$.

Proof: Set $\xi = (1+\alpha)/2$ and $m = \xi n$. Let Y be an m-dimensional quotient of X and let $|\cdot|_2$ be the Euclidean norm on Y as in Proposition 3.5. Set $\alpha_1 = \alpha/\xi$, $\beta_1 = \beta/\xi$ and $\beta'_1 = \beta'/\xi$. Applying Theorem 2.6 to Y and $0 < \beta_1 < \alpha_1/2$ and $\gamma = \min\left((\alpha_1/2 - \beta_1)/2, 1 - \alpha_1\right)$ we get

$$d(F_1, F_2) \geq c' \tilde{V}^{-2}_{\beta_1 m, \beta'_1 m}(Y),$$

where $c' = c'(\alpha, \beta, \beta') > 0$ and $\tilde{V}_{\beta_1 m, \beta'_1 m}$ is defined by (2.1) with respect to the norm $|\cdot|_2$. Now, it suffices to observe that this combined with (3.9) and the definitions of α_1, β_1 and β'_1 imply (3.18). □

It can be shown that for $1 \leq p \leq 2$ the volume ratio numbers of spaces l^p_n, with respect to the natural Euclidean norm, satisfy

$$V_{l,k}(l^p_n) \leq V_{k,k}(l^p_n) \leq a k^{1/2 - 1/p}, \tag{3.19}$$

for $1 \leq l \leq k \leq n$, where a is a universal constant (cf. e.g [P-T] (1.3)). Therefore Theorem 3.6 yields an immediate corollary, which for $p = 1$ and $\alpha = 1/3$ is the original statement of Gluskin's theorem [G.1].

COROLLARY 3.7: *Let $1 \le p \le 2$. For every $0 < \alpha < 1$ the space l_n^p has αn-dimensional quotients, say F_p, F_p' such that*

$$d(F_p, F_p') \ge cn^{2/p-1},$$

where $c = c(\alpha) > 0$.

4. A Solution of the Finite-Dimensional Version of the Homogeneous Spaces Problem and Related Results

To make the results of this section more transparent and to avoid repetitions, for every n-dimensional Banach space X and for every $1 \le k < n$ set

$$K(X,k) = \sup\{ \text{ d } (F_1, F_2)\}, \tag{4.1}$$

where the supremum is taken over all pairs of k-dimensional quotients of X. Similarly, define $K^*(X,k)$ to be the diameter of the set of all k-dimensional subspaces of X. It is obvious by the duality that $K(X,k) = K^*(X^*,k)$. The following result is a solution of the finite-dimensional homogeneous spaces problem.

THEOREM 4.1: *Let $0 < \alpha < 1$ and let X be an n-dimensional Banach space. Let $K = \min(K(X,\alpha n), K^*(X,\alpha n))$. Then*

$$d(X, l_n^2) \le \begin{cases} cK^2 & \text{if } 2/3 \le \alpha < 1, \\ cK^{3/2} & \text{if } 0 < \alpha < 2/3, \end{cases}$$

where $c = c(\alpha)$.

The proof of the theorem requires the following proposition, which is based on an argument by Milman and Pisier [M-P.1] (cf. also [B.2]).

THEOREM 4.2: *Let $0 < \delta < \xi < 1$. Let Z be an n-dimensional space such that every ξn-dimensional subspace Z_1 of Z contains a subspace H with $\dim H = \delta n$ such that*

$$d(H, l_{\dim H}^2) \le D,$$

for some $D \ge 1$. Then for every $0 < \eta < 1 + \delta - \xi$ there exists a subspace S of Z with $\dim S = (1 + \delta - \xi - \eta)n$ such that

$$d(S, l_{\dim S}^2) \le cD,$$

where $c = c(\xi, \delta, \eta)$.

Proof: Clearly it is enough to prove the proposition for η sufficiently small. Therefore fix $0 < \eta < 2\min(\delta, 1-\xi)$ (and so, smaller than $1+\delta-\xi$). Let $u : Z \to l_n^2$ be the isomorphism constructed in Theorem 3.3 which satisfies (3.6). Set $u^{-1} = w$. Since $c_{\eta n/2}(u) \le a/\eta$, where a is a universal constant, there is a $(1-\eta/2)n$-dimensional subspace $F \subset Z$ such that

$$\|u \,|\, F : F \to l_n^2\| \le a/\eta. \tag{4.2}$$

Let $Y_1 \subset F$ be a subspace with $\dim Y_1 = \delta n$ such that $\mathrm{d}(Y_1, l_{\delta n}^2) \le D$. Set $G_1 = u(Y_1) \subset l_n^2$ and let $w_1 : G_1 \to Y_1$ be the restriction of w. Then

$$d_m(w_1) \le \tilde{d}_m(w) \le a(n/m)^2,$$

for every $1 \le m \le n$. Observe that w_1 is an operator acting between spaces D-isomorphic to Hilbert spaces and that for such operators the Kolmogorov numbers and the Gelfand numbers coincide (up to a constant D). In particular, $c_m(w_1) \le D\, d_m(w_1)$. This yields the existence of a subspace $H_1 \subset G_1$ with $\dim H_1 = (\delta - \eta/2)n$ such that

$$\|w \,|\, H_1 : H_1 \to F\| = \|w_1 \,|\, H_1 : H_1 \to Y_1\| \le cD,$$

where $c = c(\delta, \eta)$.

The above argument leads to the inductive construction of mutually orthogonal subspaces $H_1, H_2, \ldots$ of l_n^2 and subspaces $F_0 = F \supset F_1 \supset \cdots$ such that $w(H_j) \subset F_{j-1}$,

$$\|w \,|\, H_j : H_j \to F\| \le cD,$$

and $\dim H_j = (\delta - \eta/2)n$, for $j = 1, 2, \ldots$. In the s-th step set

$$F_s = w\Big(\big(\bigoplus_{j=1}^{s} H_j\big)^{\perp}\Big) \cap F \text{ and let } l_s = \dim F_s = \dim F - \dim \bigoplus_{j=1}^{s} H_j,$$

for $s = 0, 1, 2, \ldots$. Consider the least integer k such that $l_k < \xi n$. Since $l_{k-1} \ge \xi n$ then

$$k \le 4/\delta.$$

If $l_k \le (\xi - (\delta - \eta/2))n$, let $H_{k+1} = \{0\}$ and end the construction. Otherwise, decrease the dimension of the space H_k to have $l_k = \xi n$. Using the argument above for the last time construct a subspace $H_{k+1} \subset (\bigoplus_{j=1}^{k} H_j)^{\perp}$ such that

$$\|w \,|\, H_{k+1} : H_{k+1} \to F\| \le cD$$

and $\dim H_{k+1} = (\delta - \eta/2)n$.

Set

$$S = \bigoplus_{j=1}^{k+1} H_j.$$

Observe that

$$\dim S \geq \dim F - (\xi - (\delta - \eta/2))n = (1 + \delta - \eta - \xi)n.$$

Moreover,

$$\begin{aligned} \|w \mid S : S \to F\| &\leq \left(\sum_{j=1}^{k+1} \|w \mid H_j : H_j \to F\|^2 \right)^{1/2} \\ &\leq (k+1)^{1/2} cD \leq (c'/\sqrt{\delta})D. \end{aligned}$$

Combining the last inequality with (4.2) we get

$$\mathrm{d}(S, l^2_{\dim S}) \leq c'' D.$$

This completes the proof. □

Now we are ready for the

Proof of Theorem 4.1: Since $\mathrm{d}(X, l_n^2) = \mathrm{d}(X^*, l_n^2)$, it is clearly enough to prove the theorem in the "quotient setting", $K = K(X, \alpha n)$. Assume first that $2/3 \leq \alpha < 1$.

Set $\varepsilon = (1-\alpha)/6$. Set $\xi = 1 - 4\varepsilon$ and $\delta = \alpha - 3\varepsilon$, so that $0 < \delta < \alpha < \xi < 1$ and $1 + \delta - \xi = \alpha + \varepsilon > \alpha$. Fix an arbitrary quotient space X_1 of X with $\dim X_1 = \xi n$. By passing several times to suitable quotients of X_1 we will show that X_1 admits a δn-dimensional well-Euclidean quotient.

Set $\lambda = 1 - 5\varepsilon$. Let Y be a quotient space of X_1 with $\dim Y = m = \lambda n$, as constructed in Proposition 3.5. Let $|\cdot|_2$ be the suitable Euclidean norm on Y, and let B_2 denotes its unit ball.

Set $\alpha_1 = \alpha/\lambda$ (note that $\alpha_1 < 1$). Let $\beta = \varepsilon/\lambda$ and $\beta' = (\alpha - \varepsilon)/\lambda$, so that $0 < \beta < \beta' < \alpha_1 < 1$. Let $\gamma = \min\left((\alpha_1 - \beta')/4, 1 - \alpha_1\right)$. Applying Theorem 2.8 to Y, and combining the definition of K, (3.8), (2.14) and Proposition 3.2 we get

$$K \geq c' V^{-1}_{\beta m, \beta' m}(Y), \tag{4.3}$$

where $c' = c'(\alpha) > 0$.

By (2.4), there exists a quotient Y_1 of Y, with $\dim Y_1 = \beta' m$, such that

$$V^{-1}_{\beta m, \beta' m}(Y) = \mathrm{vr}_{\beta m}(i_{2,Y_1}),$$

where $i_{2,Y_1} : (Y_1, |\cdot|_2) \to Y_1$ is the formal identity operator and $|\cdot|_2$ denotes the Euclidean norm on Y_1 induced from Y. Set $\delta_1 = \delta/\lambda$. By Proposition 3.4 we get

$$d_{2\beta m}(i_{2,Y_1}) \le a \ \mathrm{vr}_{\beta m}(i_{2,Y_1}) \le C\,K,$$

where $a > 1$ is a universal constant and $C = C(\alpha)$. Since $\beta' m - 2\beta m = \delta_1 m$, this means that there exists a $\delta_1 m$-dimensional quotient G of Y_1, of the form $G = P'(Y_1) = P(Y)$, for some orthogonal projections P' and P, and with the (quotient) norm $\|\cdot\|_G$, such that

$$\|z\|_G \le C\,K|z|_2 \quad \text{for} \quad z \in G.$$

By (3.7), $|y|_2 \le \|y\|$ for $y \in Y$, and hence also $|z|_2 \le \|z\|_G$ for $z \in G$ (by the definition (2.1) of the quotient norm). Therefore

$$\mathrm{d}(G, l^2_{\delta_1 m}) \le C\,K,$$

where $C = C(\alpha)$.

The above argument shows by duality that every ξn-dimensional subspace $Z = (X_1)^*$ of X^* contains a subspace $H = G^*$ with $\dim H = \delta n$ such that

$$\mathrm{d}(H, l^2_{\dim H}) \le C\,K,$$

where $C = C(\alpha)$. Now use Proposition 4.2 with $\eta = \varepsilon$. Let S be a subspace of X^* such that

$$\mathrm{d}(S, l^2_{\dim S}) \le C' K, \tag{4.4}$$

where $C' = C'(\alpha)$ and

$$\dim S \ge (1 + \delta - \xi - \eta)n = \alpha n.$$

Therefore every αn-dimensional subspace Z_1 of X^* satisfies

$$\mathrm{d}(Z_1, l^2_{\alpha n}) \le K^*(X^*, \alpha n)\ \mathrm{d}(S, l^2_{\dim S}) \le C' K^2.$$

By [T] Proposition 26.2 the last estimate yields

$$\mathrm{d}(X, l^2_n) = \mathrm{d}(X^*, l^2_n) \le (1/\alpha)\ \mathrm{d}(Z_1, l^2_{\dim Z_1}) \le C'' K^2,$$

where $C'' = C''(\alpha)$. This concludes the proof in the case $2/3 \le \alpha < 1$.

In the remaining case, when $0 < \alpha < 2/3$, we let $\varepsilon = \min((1-3\alpha/2)/6, \alpha/12)$. Set $\xi = 1-\alpha/2-4\varepsilon$ and $\delta = \alpha/2-3\varepsilon$. Then let $\lambda = 1-\alpha/2-5\varepsilon$ and let $\beta = \varepsilon\lambda$ and $\beta' = (\alpha/2-\varepsilon)/\lambda$. Now the proof goes exactly along the same lines, except that we use Theorem 2.6 instead of Theorem 2.8. □

A restatement of a particular case considered above establishes a lower estimate for the distance between subspaces or quotient spaces of a given finite-dimensional space in terms of the Euclidean distance of the space, thus complementing Theorem 3.6.

COROLLARY 4.3: *Let $0 < \alpha < 2/3$ and let X be an n-dimensional Banach space. Let $K = \min(K(X, \alpha n), K^*(X, \alpha n))$. Then*

$$K \geq c\, d(X, l_n^2)^{2/3},$$

where $c = c(\alpha) > 0$.

It would be interesting to improve the above estimate to have $K \geq c\, d(Y, l_n^2)$, at least for some $0 < \alpha < 2/3$. There are easy examples that the exponent 1 on the right hand side of this inequality is best possible.

The following standard consequence of Theorem 4.1 is worth noting (cf. e.g. Proposition 26.1 in [T]).

COROLLARY 4.4: *Let $0 < \alpha < 1$, let $1 \leq k \leq \alpha n$ and let Y be an n-dimensional Banach space. Let $K = \min(K(Y, k), K^*(Y, k))$. Then*

$$d(Y, l_n^2) \leq C\, K^{2b \log n / \log k},$$

where $b = 3/2$, if $0 < \alpha < 2/3$ and $b = 2$, if $2/3 \leq \alpha < 1$.

It easily follows from the proof of Theorem 3.6 that if X has bounded volume ratio, then (3.18) holds for random quotients F_1, F_2 of X. This is, in particular, the situation of Corollary 3.7, when F_p, F_p' are random quotients of l_n^p. On the other hand it seems that for an arbitrary Banach space X, random quotients of X do not necessarily satisfy (3.18). Therefore, in general case, it seems natural to consider random quotients of a (non-random) quotient of X rather than random quotients of X itself. In such a context one can prove the following interesting result.

THEOREM 4.5: *Let $0 < \alpha < 2/3$. Let X be an n-dimensional Banach space, and let $\|\cdot\|_2$ be the Euclidean norm on X determined by the ellipsoid of minimal volume containing the unit ball of X. There exists a quotient X_1 of X with $\dim X_1 > (3/2)\alpha n$ such that for every $A \geq 1$ at least one of the following two conditions holds.*

(i)

Random αn-dimensional quotients F_1, F_2 of X_1 satisfy

$$d(F_1, F_2) \geq A. \tag{4.5}$$

(ii)

One has $d(X_1, l^2_{\dim X_1}) \leq cA^{1/2}$. Moreover, a random αn-dimensional quotient F of X_1 satisfies

$$(1/c_1)\|x\|_2 \leq \|x\|_F \leq c_2\|x\|_2 \quad \text{for} \quad x \in F, \tag{4.6}$$

where $c = c(\alpha)$ and $c_1c_2 = c\,A^{1/2}$.

For the proof of this result we refer the reader to [M-T.2] (cf. also [M-T.1]). Let us note that if α is small enough then $c(\alpha)$ can be taken smaller than 1, making the conditions (i) and (ii) exclusive.

5. Proof of Proposition 2.2

We present the postponed proof of Proposition 2.2.

Proof: Denote the unit ball $Q_E(B_n)$ in $Y_{\alpha n}$ by $B_{\alpha n}$. To describe both cases (i) and (ii) simultaneously, set η equal to 2γ in case (i) or to $\alpha/2$ in case (ii). Fix an operator $T \in L(\mathbb{R}^{\alpha n})$ satisfying the requirements of the proposition and observe that without loss of generality we may and shall assume that all s-numbers of T are distinct. Replacing T by ξT for a suitable choice of $\xi < 1$ we may assume that T has exactly ηn s-numbers smaller than 1.

Fix $F_0 \in \mathcal{F}$. Remember that $\gamma \leq 1 - \alpha$ and observe that we clearly have

$$\begin{aligned} &h_{n,(1-\alpha)n}(\{E \in \mathcal{F} \mid \|TQ_E : l^1_n \to Y_{\alpha n}\| \leq A\}) \\ &\leq h_n(\{U \in \mathcal{O}(n) \mid U(F_0) \in \mathcal{F},\ TQ_{U(F_0)}e_i \in AB_{\alpha n} \\ &\qquad \text{for } (1-\gamma)n < i \leq n\}). \end{aligned} \tag{5.1}$$

By $\mathcal{H}$ denote the group

$$\mathcal{H} = \{V \in \mathcal{O}(n) \quad | \quad V\,|\,(\mathbb{R}^{\alpha n})^{\perp} = \text{Id}\}. \tag{5.2}$$

Clearly $\mathcal{H}$ is isomorphic to $\mathcal{O}(\alpha n)$. For $k = 0, 1, \dots, \gamma n - 1$ and $V \in \mathcal{H}$ set

$$\mathcal{A}_k \otimes V = \{U \in \mathcal{O}(n) \mid U(F_0) \in \mathcal{F},\ TQ_{VU(F_0)}e_i \in AB_{\alpha n} \quad \text{for} \quad (1-\gamma)n < i \leq n-k\},$$

and

$$\mathcal{A}^k \otimes V = \{U \in \mathcal{O}(n) \mid U(F_0) \in \mathcal{F},\ TQ_{VU(F_0)}e_{n-k} \in AB_{\alpha n}\}.$$

Before we go further let us note that

$$Q_{VU(F_0)} = VQ_{U(F_0)}V^{-1}, \tag{5.3}$$

for every $V \in \mathcal{H}$ and $U \in \mathcal{O}(n)$. Indeed, it is enough to check this equality separately on $\mathbb{R}^{\alpha n}$ and $VU(F_0)$ (since $\mathbb{R}^n$ is a direct sum of these subspaces). The set on the right hand side of (5.1) is $\mathcal{A}_0 \otimes \text{ Id}$. From Fact III it follows that for every $V \in \mathcal{H}$ we have

$$h_n(\mathcal{A}_k \otimes \text{ Id}) = \int_{\mathcal{O}(n)} \chi_{\mathcal{A}_k \otimes \text{ Id}}\, dU = \int_{\mathcal{O}(n)} \chi_{\mathcal{A}_k \otimes V}\, dU.$$

Integrating over $V \in \mathcal{H}$ we get by Fubini's theorem

$$h_n(\mathcal{A}_k \otimes \text{ Id}) = \int_{\mathcal{O}(n)} \left(\int_{\mathcal{H}} \chi_{\mathcal{A}_k \otimes V}\, d_{\mathcal{H}}V \right) dU \tag{5.4}$$

for $k = 0, 1, \dots, \gamma n - 1$.

We will prove that the following inequality holds for all $U \in \mathcal{O}(n)$ such that $U(F_0) \in \mathcal{F}$ and for $k = 0, 1, \dots, \gamma n - 1$,

$$\int_{\mathcal{H}} \chi_{\mathcal{A}_k \otimes V}\, d_{\mathcal{H}}V \leq \left(c\varepsilon^{-1} AV_{\beta n, \beta' n}(X)\right)^{\beta n} \int_{\mathcal{H}} \chi_{\mathcal{A}_{k+1} \otimes V}\, d_{\mathcal{H}}V, \tag{5.5}$$

where ε satisfies condition (C) and $c = c(\alpha, \beta)$.

Assuming (5.5) we can easily complete the proof of Proposition 2.2 as follows. By (5.4) and (5.5) we get

$$\begin{aligned} h_n(\mathcal{A}_0 \otimes \text{ Id}) &\leq \left(c\varepsilon^{-1} AV_{\beta n, \beta' n}(X)\right)^{\beta n} \int_{\mathcal{O}(n)} \left(\int_{\mathcal{H}} \chi_{\mathcal{A}_1 \otimes TV}\, d_{\mathcal{H}}V \right) dU \\ &= \left(c\varepsilon^{-1} AV_{\beta n, \beta' n}(X)\right)^{\beta n} h_n(\mathcal{A}_1 \otimes \text{ Id}). \end{aligned}$$

Repeating this argument $\gamma n - 1$ times we get

$$h_n(\mathcal{A}_0 \otimes \text{ Id}) \leq (c\varepsilon^{-1} AV_{\beta n, \beta' n}(X))^{\gamma \beta n^2} h_n(\mathcal{A}_{\gamma n}).$$

Since $h_n(\mathcal{A}_{\gamma n}) \leq 1$, Proposition 2.2 would follow directly from (5.1) and the definition of $\mathcal{A}_0 \otimes$ Id.

To show (5.5) first observe that for every $k = 0, \dots, \gamma n - 1$ and every $V \in \mathcal{H}$ we have

$$\chi_{\mathcal{A}_k \otimes V} = \chi_{\mathcal{A}_{k+1} \otimes V} \chi_{\mathcal{A}^k \otimes V}.$$

Therefore, for $k = 0, 1, \dots, \gamma n - 1$ and $U \in \mathcal{O}(n)$, we have

$$\int_{\mathcal{H}} \chi_{\mathcal{A}_k \otimes V} \, d_{\mathcal{H}} V = \int_{\mathcal{H}} \chi_{\mathcal{A}_{k+1} \otimes V} \chi_{\mathcal{A}^k \otimes V} \, d_{\mathcal{H}} V. \tag{5.6}$$

Fix $U \in \mathcal{O}(n)$ such that $U(F_0) \in \mathcal{F}$. In order to simplify the notation we shall prove (5.5) for $k = 0$. The case of arbitrary $k \in \{1, 2, \dots, \gamma n - 1\}$ can be proved in exactly the same way. Denote $\{e_1, \dots, e_{\gamma n - 1}\}$ by $\mathbb{R}^{\gamma n - 1}$ and observe that, by Fact III, without loss of generality we may assume that

$$Q_{U(F_0)} e_i \in \mathbb{R}^{\gamma n - 1} \quad \text{for} \quad (1 - \gamma) n < i \leq n - 1. \tag{5.7}$$

By $\mathcal{G}$ denote the group of all $W \in \mathcal{O}(n)$ such that

$$W \,|\, \mathbb{R}^{\gamma n - 1} = \text{Id} \quad \text{and} \quad W \,|\, (\mathbb{R}^{\alpha n})^{\perp} = \text{Id}. \tag{5.8}$$

Clearly $\mathcal{G}$ is isomorphic to $\mathcal{O}((\alpha - \gamma) n + 1)$. Let $h_{\mathcal{G}}$ denote the normalized Haar measure on $\mathcal{G}$. The same argument as in (5.4) and (5.6) yields that

$$\int_{\mathcal{H}} \chi_{\mathcal{A}_1 \otimes V} \chi_{\mathcal{A}^0 \otimes V} \, d_{\mathcal{H}} V = \int_{\mathcal{H}} \left(\int_{\mathcal{G}} \chi_{\mathcal{A}_1 \otimes VW} \chi_{\mathcal{A}^0 \otimes VW} \, d_{\mathcal{G}} W \right) d_{\mathcal{H}} V.$$

Since $1 - \gamma \geq \alpha$, by (5.2), (5.3), (5.7) and (5.8), we have, for $(1 - \gamma) n < i \leq n - 1$,

$$\begin{aligned} Q_{VWU(F_0)} e_i &= VW Q_{U(F_0)} W^{-1} V^{-1} e_i = VW Q_{U(F_0)} e_i \\ &= V Q_{U(F_0)} e_i = Q_{VU(F_0)} e_i, \end{aligned}$$

so the function $\chi_{\mathcal{A}_1 \otimes VW}$ does not depend on W. Therefore

$$\int_{\mathcal{H}} \chi_{\mathcal{A}_1 \otimes V} \chi_{\mathcal{A}^0 \otimes V} \, d_{\mathcal{H}} V = \int_{\mathcal{H}} \chi_{\mathcal{A}_1 \otimes V} \left(\int_{\mathcal{G}} \chi_{\mathcal{A}^0 \otimes VW} \, d_{\mathcal{G}} W \right) d_{\mathcal{H}} V. \tag{5.9}$$

Observe that by (5.8) and (5.2), $\mathcal{G} \subset \mathcal{H}$ and so, by (5.3), the inner integral in (5.9) satisfies

$$\begin{aligned} &\int_{\mathcal{G}} \chi_{\mathcal{A}^0 \otimes VW} \, d_{\mathcal{G}} W \\ &\quad = h_{\mathcal{G}} \{ W \in \mathcal{G} \,|\, U(F_0) \in \mathcal{F},\ T Q_{VWU(F_0)} e_n \in A B_{\alpha n}) \} \\ &\quad = h_{\mathcal{G}} \{ W \in \mathcal{G} \,|\, U(F_0) \in \mathcal{F},\ TVW Q_{U(F_0)} W^{-1} V^{-1} e_n \in A B_{\alpha n}) \} \\ &\quad = h_{\mathcal{G}} \{ W \in \mathcal{G} \,|\, U(F_0) \in \mathcal{F},\ TVW Q_{U(F_0)} e_n \in A B_{\alpha n}) \}. \end{aligned} \tag{5.10}$$

Since the operator TV has the same s-numbers as T has, for $V \in \mathcal{H}$, (5.5) is an immediate consequence of (5.6), (5.9), (5.10) and the following lemma.

LEMMA 5.1: *Let $0 < \alpha < 1$ and let $A > 0$. Let $E_0 \in G_{n,(1-\alpha)n}$ satisfy (A) and let $B_{\alpha n} = Q_{E_0} B_n$. Let β, β' and γ satisfy the assumptions of Proposition 2.2 case (i) or (ii), and let η be defined as at the beginning of the proof of Proposition 2.2. Let $\mathcal{G}$ be the group defined in (5.8). Then for every $F_0 \in \mathcal{F}$ and every $U \in \mathcal{O}(n)$ such that $U(F_0) \in \mathcal{F}$, we have*

$$h_{\mathcal{G}}\{W \in \mathcal{G} \,|\, TWQ_{U(F_0)}e_n \in AB_{\alpha n})\} \le (c\varepsilon^{-1} A V_{\beta n, \beta' n})^{\beta n},$$

for every $T \in L(\mathbb{R}^n)$ which has exactly ηn s-numbers smaller than 1, where ε satisfies condition (C) and $c = c(\alpha, \beta, \beta')$.

Proof: By condition (A), $E_0^\perp = E_1 \oplus E_1'$ with $E_1 \perp E_1'$, $Q_{E_0}(E_1) \perp Q_{E_0}(E_1')$ and $\dim E_1 \le \gamma n$, and such that $Q_{E_0} \,|\, E_1$ (resp. $Q_{E_0} \,|\, E_1'$) has all s-numbers larger than c_1 (resp. smaller than c_1). Set $F_1 = Q_{E_0}(E_1) \subset \mathbb{R}^{\alpha n}$.

Fix an operator $T \in L(\mathbb{R}^n)$ which has exactly ηn s-numbers smaller than 1. Write $\mathbb{R}^{\alpha n} = E_2 \oplus E_2'$ with $E_2 \perp E_2'$, $T(E_2) \perp T(E_2')$ and $\dim E_2 = \eta n$, and such that $T \,|\, E_2$ (resp. $T \,|\, E_2'$) has all s-numbers smaller than 1 (resp. larger than or equal to 1). Set $F_2 = T(E_2) \subset \mathbb{R}^{\alpha n}$.

Set $F_3 = T(\mathbb{R}^{\gamma n - 1})$, so that $\dim F_3 < \gamma n$. Observe that by (5.7) and (5.8), for every $W \in \mathcal{G}$ we have

$$TW(\,\mathrm{span}\{Q_{E_0} e_i \mid (1-\gamma)n < i \le n-1\}) \subset F_3.$$

Let $\tilde{E} \subset \mathbb{R}^{\alpha n}$ be any $\beta' n$-dimensional subspace such that

$$\tilde{E} \perp F_j \quad \text{for } j = 1, 2, 3.$$

(For $\tilde{E}$ like this to exist we need $\alpha - (\eta + 2\gamma) \ge \beta'$. This inequality in turn follows directly from the definition of η and the inequalities $\gamma \le (\alpha - \beta')/2$ in case (i) and $\gamma \le (\alpha/2 - \beta')/2$ in case (ii).)

Set $S = Q_{E_0} \,|\, E_0^\perp : E_0^\perp \to \mathbb{R}^{\alpha n}$ and $E = (S^{-1}(\tilde{E}))^\perp$. Clearly, $E \supset E_0$ and $\dim E = (1-\beta')n$. By the definition of $V_{\beta n, \beta' n} = V_{\beta n, \beta' n}(B_n)$ there exists a $(1-\beta)n$-dimensional subspace $F \supset E$ of $\mathbb{R}^n$ such that

$$(5.11) \qquad \left(\,\mathrm{vol}_{\beta n} P_F(B_n) / \,\mathrm{vol}_{\beta n} P_F(B_n^2)\right)^{1/\beta n} \le V_{\beta n, \beta' n}.$$

Observe that $SP_F \,|\, \mathbb{R}^{\alpha n}$ is a projection in $\mathbb{R}^{\alpha n}$ and $SP_F(\mathbb{R}^{\alpha n}) \subset \tilde{E}$. Let P be the orthogonal projection in $\mathbb{R}^{\alpha n}$ with the same kernel as $SP_F \,|\, \mathbb{R}^{\alpha n}$.

Set

$$\mathcal{A}' = \{W \in \mathcal{G} \mid PTWQ_{U(F_0)}e_n \in APB_{\alpha n}\}.$$

Clearly, if $\mathcal{A}$ denotes the set in the conclusion of the lemma, then

$$\mathcal{A} \subset \mathcal{A}'. \tag{5.12}$$

Set $R = T\,|\,E_2'$. Recall that R has all s-numbers larger than or equal to 1 and in particular it is invertible. Since $F_2 \subset \ker P$, the operator $P' = R^{-1}PT$ is well defined and it is a projection. In fact, $PT = TP' = RP'$, therefore,

$$\mathcal{A}' = \{W \in \mathcal{G} \mid P'WQ_{U(F_0)}e_n \in AR^{-1}PB_{\alpha n}\}.$$

Note that $\mathbf{R}^{\gamma n-1} = \operatorname{span}\{e_1, \dots, e_{\gamma n-1}\}$ satisfies $\mathbf{R}^{\gamma n-1} \subset \ker P'$.

Denote by H the orthogonal complement of $\mathbf{R}^{\gamma n-1}$ in $\mathbf{R}^{\alpha n}$ and by P_1 the orthogonal projection on H. Let P_0 be the projection in $\mathbf{R}^{\alpha n}$ such that $P' = P_0P_1$. Clearly, $WP_1 = P_1W$ for every $W \in \mathcal{G}$. Set

$$z = Q_{U(F_0)}e_n$$

and

$$z_0 = P_1(z)/\|P_1(z)\|_2.$$

Clearly $P'WQ_{U(F_0)}e_n = P_0W(z_0)\|P_1(z)\|_2$. Recall that $U(F_0) \in \mathcal{F}$ and in particular it satisfies condition (C). Let $\varepsilon > 0$ be a constant as in (C). Set

$$\mathcal{A}'' = \{W \in \mathcal{G} \mid P_0W(z_0) \in \varepsilon^{-1}AR^{-1}PB_{\alpha n}\}. \tag{5.13}$$

Then $\|P_1(z)\|_2 \geq \varepsilon$. Thus

$$\mathcal{A}' \subset \mathcal{A}''. \tag{5.14}$$

Since $\mathcal{G}$ can be identified with the orthogonal group acting on H, the measure of $\mathcal{A}''$ is equal to the measure of a suitable subset of the sphere S_H in H. We have

$$h_{\mathcal{G}}(\mathcal{A}'') = \mu_{S_H}\left(\{x \in S_H \mid P_0x \in \varepsilon^{-1}AR^{-1}PB_{\alpha n}\}\right). \tag{5.15}$$

Denote by $\tilde{A}$ the subset of S_H which appears on the right hand side of (5.15). The estimate for the measure of such a set is well known. Using e.g. Lemma 3 in [Ma.1] we get

$$\begin{aligned} \mu_H(\tilde{A}) &\leq \frac{\operatorname{vol}\,(\varepsilon^{-1}AR^{-1}PB_{\alpha n})\ \operatorname{vol}\,B_k^2}{\operatorname{vol}\,B^2_{\dim H}} \\ &\leq \left(c\varepsilon^{-1}A\right)^{\beta n}\frac{\operatorname{vol}\,R^{-1}PB_{\alpha n}}{\operatorname{vol}\,B^2_{\beta n}}, \end{aligned} \tag{5.16}$$

where $c = c(\alpha, \beta) > 0$, and $k = \dim \ker P_0 = \dim H - \beta n$.

It remains to estimate vol$R^{-1}PB_{\alpha n}$. Observe first that since all s-numbers of R are larger than or equal to 1 then R^{-1} is a contraction (in the Euclidean norm) and

$$\text{vol } R^{-1}PB_{\alpha n} \leq \text{ vol } PB_{\alpha n}.$$

Clearly,

$$\text{vol } PB_{\alpha n} \leq \text{ vol } SP_F B_{\alpha n}.$$

Observe that $SP_F(\mathbb{R}^{\alpha n}) \subset \tilde{E} \subset Q_{E_0}(E_1'))$. Moreover, by property (A), $Q_{E_0} \,|\, E_1' : E_1' \to Q_{E_0}(E_1')$ is a c_1-isomorphism (in the Euclidean norm). Also, $F \supset E_0$. Thus

$$\text{vol } SP_F B_{\alpha n} \leq c_1^{\beta n} \text{ vol } P_F B_{\alpha n} = c_1^{\beta n} \text{ vol } P_F B_n.$$

Therefore, by (5.11), we have

$$\text{vol } PB_{\alpha n} \leq (c_1 V_{\beta n, \beta' n}(X))^{\beta n} \text{ vol } B^2_{\beta n}.$$

Combining the later estimate with (5.16), (5.15), (5.14) and (5.12), we conclude the proof of Lemma 5.1 and hence the proof of Proposition 2.2.

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